NAME: ______________________ ID: _______________ DOB: __/__/19__
Department/Program: TBK / ___

Re-examination -- Mechatronics (NAMO05E)
Tuesday, August 26, 2008 (9:00-12:00)

Please write completely your name, student ID number and date of birth on this or the first page. For all subsequent pages, you only need to write your name and the page number. This is an open-book exam.
The final mark of this exam is capped to 100 (when the bonus marks are included).

Question 1. (Total mark: 25)

The iBOT® is an advance wheelchair system which (in addition to the standard wheelchair functionality) allows the user to climb the stairs and to balance itself on two wheels during its upright position (see Figure 1).

Figure 1. An iBOT® system.

This system is a mechatronics system which enhances the conventional wheelchair by using advanced information processing.

a) Referring to the block diagram in Figure 2, identify possible measured variables, manipulated variables and reference variables in the system. (10 marks).
b) Also based on the schematic in Figure 2, identify possible sensors and actuators that can be used. (10 marks).
c) What would be the man/machine interface in this system. (5 marks)

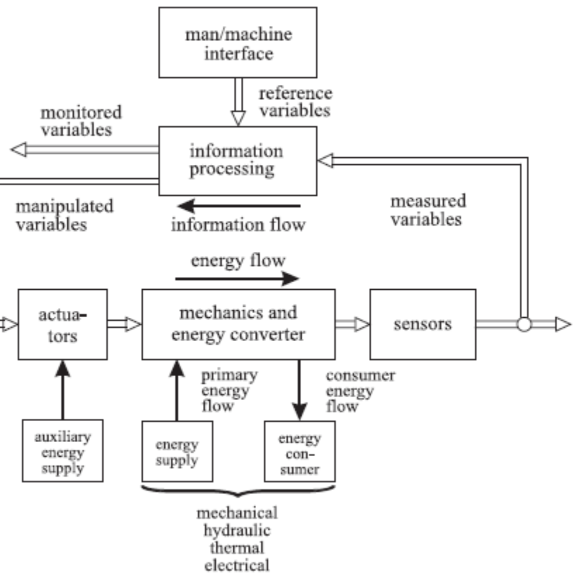

Figure 2. Basic mechatronics block diagram.

Answers.

a). Possible measured variables:

- angular position of the wheels
- angular velocity of the wheels
- angular position of the chair
- angular velocity of the chair
- electrical current in the motor
- voltage in the motor
- valve position in the hydraulic
- pressure in the hydraulic systems
- volume rate in the hydraulic systems
- the weight of the person

Possible manipulated variables:

- voltage in the motor
- current in the motor
- voltage / current to the valve mechanism in the hydraulic systems

Possible reference variables:

- angular position of the chair
- angular velocity of the chair
- angular position of the wheel
- angular velocity of the wheel

b). Sensors:
For angular position: optical encoder, hall sensors, inductive sensor
For angular velocity: tachogenerator, hall sensors, inductive sensor
For pressure: strain gauge, piezoresistive sensor, other pressure sensor devices
For volume rate: pitot tube, venturi meter, other flowrate sensor devices

c). The man/machine interface:
Joystick which determines the desirable action of the wheel chair.

Question 2. (Total mark: 25)

Let us consider the pendulum system as shown in Figure 3. The transfer function of the linearized system around the equilibrium point ($\theta = \pi/3$, $\dot{\theta} = 0$) is given by

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 - \frac{g}{2L}}.$$

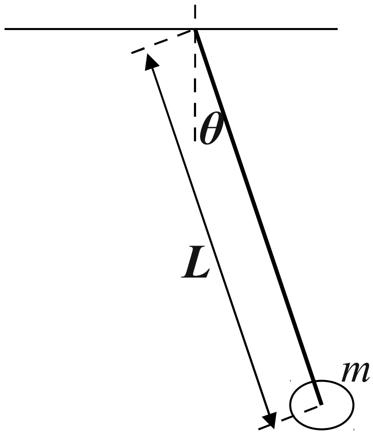

Figure 3. Simple pendulum.

The gravitational constant and the pendulum length are positive constants.

(a). Evaluate whether the linearized system can be stabilized using Proportional controller only (C(s) = K_p).

(10 marks)

(b). Suppose that we would use the Proportional + Derivative controller (C(s) = K_p + K_ds) where K_p and K_d are the controller gains, find the conditions on K_p and K_d such that the closed loop system (of the linearized system) is stable.

(15 marks)

Bonus question. Suppose that the gravitational constant g is between 9 to 10 and the length L is between 1 to 2. Using the Proportional + Derivative controller, determine the stabilizing gain K_p and K_d that can deal with the uncertainties of g and L. (The inequalities for K_p and K_d should be in numbers.) (Bonus: 10 marks)

Answers.

a). The sensitivity transfer function is:

$$\frac{1}{1+G(s)C(s)} = \frac{s^2 - \frac{g}{2L}}{s^2 - \frac{g}{2L} + K_p}$$

By using Routh-Hurwitz stability test, we need the following:

- All coefficients in χ must be positive and non-zero. Since the coefficient corresponding to s is equal to zero, and any selection of Kp cannot make the coefficient correspond to s to be positive, then the system cannot be stabilized by Proportional controller.
- The checking of Routh array is unnecessary in this case.

b). The sensitivity transfer function is:

$$\frac{1}{1+G(s)C(s)} = \frac{\left(s^2 - \frac{g}{2L}\right)}{s^2 - \frac{g}{2L} + K_p + K_d s}$$

By using Routh-Hurwitz stability test, we need the following:

- All coefficients in χ must be positive and non-zero. This implies that $(K_p - \frac{g}{2L}) > 0$, $K_d > 0$. Hence by choosing Kp > g/2L and $K_d > 0$ the positivity of coefficients in χ is guaranteed.
- The first column of Routh array must not changed sign. The Routh array for χ is given by

1	K_p-g/2L
K_d	0
K_p-g/2L	0

The first column of the array is positive if K_p > g/2L and K_d > 0.

Question 3. (Total mark: 25)

Consider the robotic manipulator shown in Figure 4. The cylinder can rotate using the external torque located at the base. The beam inside the cylinder is able to move translationally along the radial axis via an external force (hydraulic). The robot shown in Figure 4 is called Polar Robot.

Figure 4. Polar robot.

Figure 5 depicts the diagram of the corresponding robot in the horizontal plane (i.e., we consider only the 2 dimensional movement). The variable θ is the angle of cylinder and r is the distance between centre of mass of both parts. The mass of the cylinder is M and the mass of the beam is m. The moment of inertia (at the centre of mass) of the cylinder is J_1 and the moment of inertia of the beam is J_2.

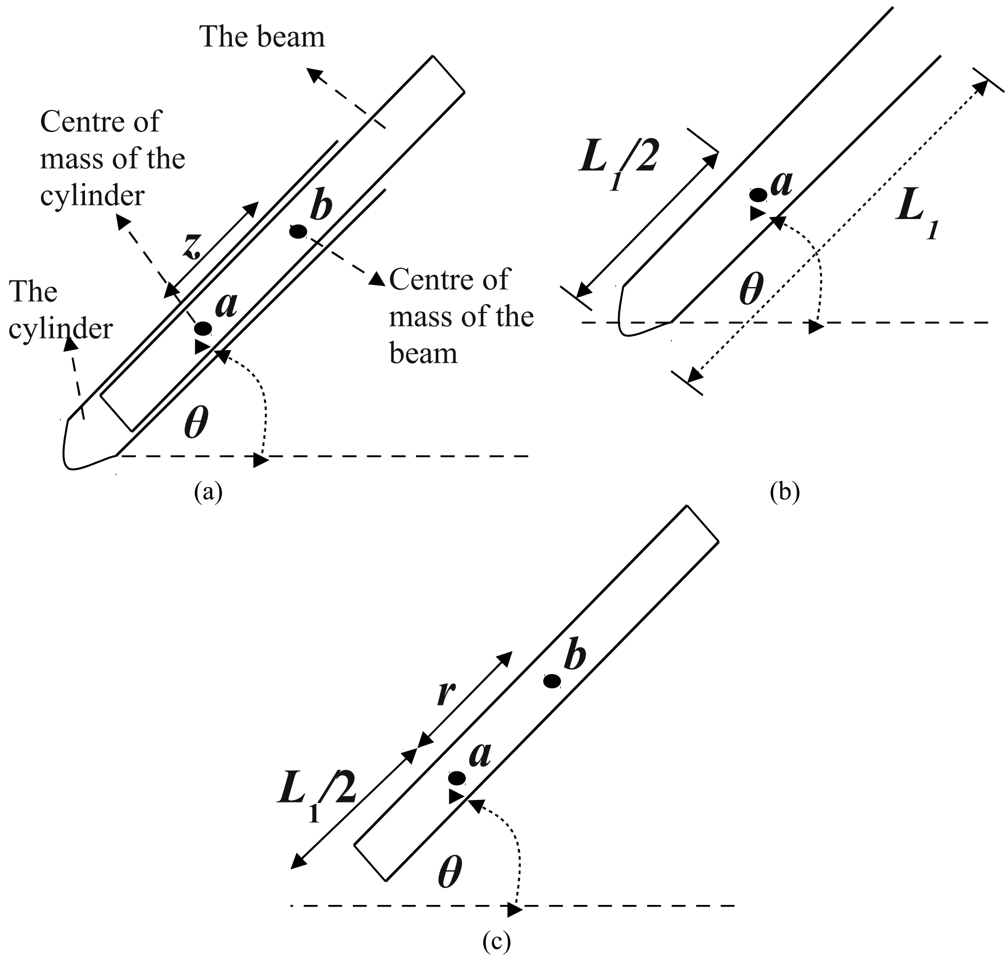

Figure 5. Diagram of polar robot: (a). The whole system, (b). The cylinder part, (c). The beam part.

The coordinate of the centre of mass of cylinder is ($x_1 = \frac{L_1}{2}\cos(\theta),\ y_1 = \frac{L_1}{2}\sin(\theta)$) and the coordinate of the centre of mass of beam is $x_2 = \left(r + \frac{L_1}{2}\right)\cos(\theta),\ y_2 = \left(r + \frac{L_1}{2}\right)\sin(\theta)$. Note that $\dot{x}_1 = -\frac{L_1}{2}\sin(\theta)\dot{\theta}$, $\dot{y}_1 = \frac{L_1}{2}\cos(\theta)\dot{\theta}$, $\dot{x}_2 = \dot{r}\cos(\theta) - \left(r + \frac{L_1}{2}\right)\sin(\theta)\dot{\theta}$ and $\dot{y}_2 = \dot{r}\sin(\theta) + \left(r + \frac{L_1}{2}\right)\cos(\theta)\dot{\theta}$.

It is assumed that there is no gravitational force affecting the system.

(a). Show that the kinetic energy of the polar robot is given by

$$E_k = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\left(M\frac{L_1^{\,2}}{4} + J_1 + J_2 + m\left(r + \frac{L_1}{2}\right)^2\right)\dot{\theta}^2$$

(10 marks)

(Hints: The kinetic energy includes the **translational kinetic energy** as well as the **rotational kinetic energy**).

(b). Using the Euler-Lagrange equations with the above kinetic energy, with the generalized coordinate $q = \begin{bmatrix}\theta \\ r\end{bmatrix}$ and with the generalized external force $F_{ext} = \begin{bmatrix}T \\ F\end{bmatrix}$

where T is the external torque and F is the external force, compute the equations of motion of the system. (Note that the potential energy is 0). (15 marks)

Answers.

a). The kinetic energy is given by:

$$E_k = \frac{1}{2}M\left(\frac{dx_1}{dt}\right)^2 + \frac{1}{2}M\left(\frac{dy_1}{dt}\right)^2 + \frac{1}{2}m\left(\frac{dx_2}{dt}\right)^2 + \frac{1}{2}m\left(\frac{dy_2}{dt}\right)^2 + \frac{1}{2}J_1\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}J_2\left(\frac{d\theta}{dt}\right)^2$$

Substituting $\dot{x}_1 = -\frac{L_1}{2}\sin(\theta)\dot{\theta}$, $\dot{y}_1 = \frac{L_1}{2}\cos(\theta)\dot{\theta}$, $\dot{x}_2 = \dot{r}\cos(\theta) - \left(r + \frac{L_1}{2}\right)\sin(\theta)\dot{\theta}$ and

$\dot{y}_2 = \dot{r}\sin(\theta) + \left(r + \frac{L_1}{2}\right)\cos(\theta)\dot{\theta}$ to the above equation, we obtain:

$$E_k = \frac{1}{2}M\left(\frac{L_1^2}{4}\sin^2(\theta)\dot{\theta}^2 + \frac{L_1^2}{4}\cos^2(\theta)\dot{\theta}^2\right) + \frac{1}{2}m\left(\dot{r}\cos(\theta) - \left(r + \frac{L_1}{2}\right)\sin(\theta)\dot{\theta}\right)^2$$

$$+ \frac{1}{2}m\left(\dot{r}\sin(\theta) + \left(r + \frac{L_1}{2}\right)\cos(\theta)\dot{\theta}\right)^2 + \frac{1}{2}J_1\dot{\theta}^2 + \frac{1}{2}J_2\dot{\theta}^2$$

$$= \frac{1}{2}M\frac{L_1^2}{4}\dot{\theta}^2 + \frac{1}{2}m\left(\dot{r}^2\cos^2(\theta) - \left(r + \frac{L_1}{2}\right)\cos(\theta)\sin(\theta)\dot{r}\dot{\theta} + \left(r + \frac{L_1}{2}\right)^2\sin^2(\theta)\dot{\theta}^2\right)$$

$$+ \frac{1}{2}m\left(\dot{r}^2\sin^2(\theta) + \left(r + \frac{L_1}{2}\right)\cos(\theta)\sin(\theta)\dot{r}\dot{\theta} + \left(r + \frac{L_1}{2}\right)^2\cos^2(\theta)\dot{\theta}^2\right) + \frac{1}{2}J_1\dot{\theta}^2 + \frac{1}{2}J_2\dot{\theta}^2$$

$$= \frac{1}{2}M\frac{L_1^2}{4}\dot{\theta}^2 + \frac{1}{2}m\left(\dot{r}^2 + \left(r + \frac{L_1}{2}\right)^2\dot{\theta}^2\right) + \frac{1}{2}J_1\dot{\theta}^2 + \frac{1}{2}J_2\dot{\theta}^2$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\left(M\frac{L_1^2}{4} + m\left(r + \frac{L_1}{2}\right)^2 + J_1 + J_2\right)\dot{\theta}^2$$

b). In order to compute the equation of motions via Euler-Lagrange, we first compute $\frac{\partial E_k}{\partial \dot{\theta}}, \frac{\partial E_k}{\partial \dot{r}}, \frac{\partial E_k}{\partial \theta}$ and $\frac{\partial E_k}{\partial r}$.

From the expression of E_k in (a), we get

$$\frac{\partial E_k}{\partial \dot{\theta}} = \left(M\frac{L_1^2}{4} + J_1 + J_2 + m\left(r + \frac{L_1}{2}\right)^2\right)\dot{\theta}$$

$$\frac{\partial E_k}{\partial \dot{r}} = m\dot{r}$$

$$\frac{\partial E_k}{\partial \theta} = 0$$

$$\frac{\partial E_k}{\partial r} = m\left(r + \frac{L_1}{2}\right)\dot{\theta}^2$$

Based on the above equations, we can now compute the equation of motions. The first one:

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\theta}}\right)-\frac{\partial E_k}{\partial \theta}=F_{ext,1}=T$$

$$\Leftrightarrow \frac{d}{dt}\left[\left(M\frac{L_1^{\,2}}{4}+J_1+J_2+m\left(r+\frac{L_1}{2}\right)^2\right)\dot{\theta}\right]=T$$

$$\Leftrightarrow \left(M\frac{L_1^{\,2}}{4}+J_1+J_2+m\left(r+\frac{L_1}{2}\right)^2\right)\ddot{\theta}+2m\left(r+\frac{L_1}{2}\right)\dot{r}\dot{\theta}=T$$

The second one:

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{r}}\right)-\frac{\partial E_k}{\partial r}=F_{ext,2}=F$$

$$\Leftrightarrow \frac{d}{dt}(m\dot{r})-m\left(r+\frac{L_1}{2}\right)\dot{\theta}^2=F$$

$$\Leftrightarrow m\ddot{r}-m\left(r+\frac{L_1}{2}\right)\dot{\theta}^2=F$$

Question 4. (Total mark: 25)

Consider again the polar robot as described in Question 3. We would like to use the Newtonian approach to derive the equations of motion of the system. Figure 6 shows the forces that contribute to the movement of each part. It is assumed that the point of contact between the cylinder and the beam is located at the end of cylinder. All parts are considered to be rigid and there is no friction between the both parts.

$F_{R,1}$ and $F_{R,2}$ are reaction forces from the base, $F_{R,3}$ is reaction force that occurs from the interaction between both parts. T is the torque that is applied at the base of the cylinder and F is the radial force that is applied to the beam.

The variable θ is the angle of the cylinder and r is the distance between centre of mass of both parts. The mass of the cylinder is M and the mass of the beam is m. The moment of inertia (at the centre of mass) of the cylinder is J_1 and the moment of inertia of the beam is J_2.

The coordinate of the centre of mass of cylinder is ($x_1=\frac{L_1}{2}\cos(\theta),\ y_1=\frac{L_1}{2}\sin(\theta)$) and the coordinate of the centre of mass of the beam is

$$x_2=\left(r+\frac{L_1}{2}\right)\cos(\theta),\ y_2=\left(r+\frac{L_1}{2}\right)\sin(\theta).$$

It is assumed that there is no gravitational force affecting the system. Using these coordinate system, we have

$$\ddot{x}_1=-\frac{L_1}{2}\cos(\theta)\dot{\theta}^2-\frac{L_1}{2}\sin(\theta)\ddot{\theta},$$

$$\ddot{y}_1=-\frac{L_1}{2}\sin(\theta)\dot{\theta}^2+\frac{L_1}{2}\cos(\theta)\ddot{\theta}$$

$$\ddot{x}_2=\ddot{r}\cos(\theta)-2\sin(\theta)\dot{r}\dot{\theta}-\left(r+\frac{L_1}{2}\right)\cos(\theta)\dot{\theta}^2-\left(r+\frac{L_1}{2}\right)\sin(\theta)\ddot{\theta}$$

$$\ddot{y}_1 = \ddot{r}\sin(\theta) + 2\cos(\theta)\dot{r}\dot{\theta} - \left(r + \frac{L_1}{2}\right)\sin(\theta)\dot{\theta}^2 + \left(r + \frac{L_1}{2}\right)\cos(\theta)\ddot{\theta}$$

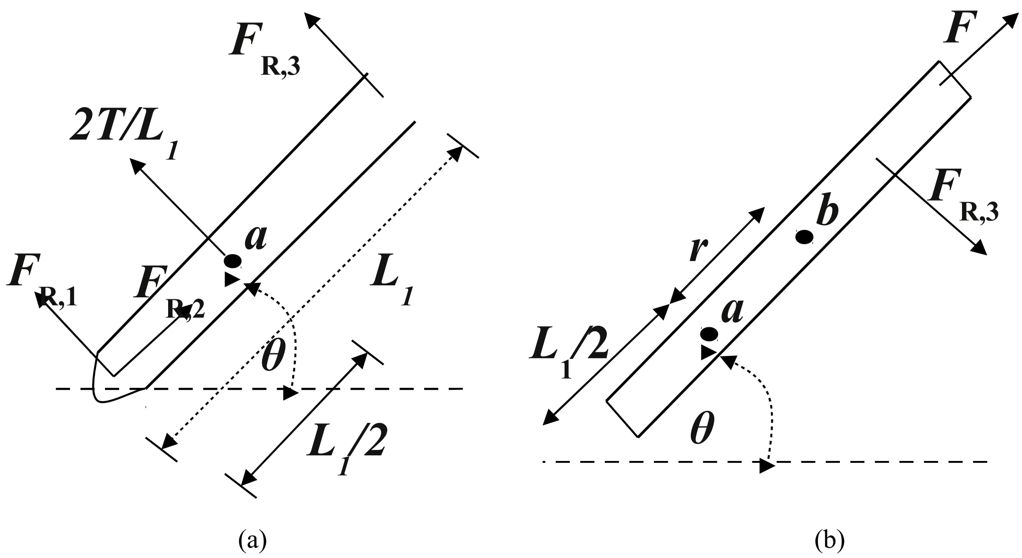

Figure 6. Diagram of the forces in the polar robot: (a). The cylinder part, (b). The beam part.

(a). Define the horizontal and vertical Newton's laws for the cylinder (at its centre of mass). (5 marks)
(b). Define the horizontal and vertical Newton's laws for the beam (at its centre of mass). (5 marks)
(c). Define the rotational Newton's law for the cylinder and the rotational Newton's law for the beam. (5 marks)
(d). Using the six equations from part (a), (b) and (c), show that the equation of motions of the polar robot is given by

$$\begin{bmatrix} M\frac{L_1^{\,2}}{4} + J_1 + J_2 + m\left(r + \frac{L_1}{2}\right)^2 & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{r} \end{bmatrix} + \begin{bmatrix} m\left(r + \frac{L_1}{2}\right)\dot{r} & m\left(r + \frac{L_1}{2}\right)\dot{\theta} \\ -m\left(r + \frac{L_1}{2}\right)\dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} T \\ F \end{bmatrix}$$

(10 marks)

(Hint: The second equation is obtained by multiplying both sides of the horizontal Newton's law of **the beam** by cos(θ), by multiplying both sides of the vertical Newton's law of **the beam** by sin(θ), and then adding the resulting equations together. To get the first equation, we need to extract three equations:
A). Multiplying both sides of the horizontal Newton's law of **the cylinder** by -sin (θ), multiplying both sides of the vertical Newton's law of **the cylinder** by cos(θ), and then adding the resulting equations together.
B). Multiplying both sides of the horizontal Newton's law of **the beam** by -sin (θ), multiplying both sides of the vertical Newton's law of **the beam** by cos(θ), and then adding the resulting equations together.
C). Combining the two rotational Newton's law for **the cylinder** and **the beam** by adding them.
Then, substitute C) to A) in order to eliminate $F_{R,1}$. The result is then substituted back to B) to get the final solution.

Bonus question. Define the state equations of the polar robot based on the above equations of motion. (Bonus: 10 marks)

Answer:
(a) The horizontal and vertical Newton's laws for the cylinder (at its centre of mass) are given as follows:

$$M\ddot{x}_1 = F_{R,2}\cos(\Theta) - \frac{2T}{L_1}\sin(\Theta) - F_{R,1}\sin(\Theta) - F_{R,3}\sin(\Theta)$$

$$M\ddot{y}_1 = F_{R,2}\sin(\Theta) + \frac{2T}{L_1}\cos(\Theta) + F_{R,1}\cos(\Theta) + F_{R,3}\cos(\Theta)$$

Substituting $\ddot{x}_1 = -\frac{L_1}{2}\cos(\theta)\dot{\theta}^2 - \frac{L_1}{2}\sin(\theta)\ddot{\theta}$ and $\ddot{y}_1 = -\frac{L_1}{2}\sin(\theta)\dot{\theta}^2 + \frac{L_1}{2}\cos(\theta)\ddot{\theta}$ to the above equations, we get

$$M\left(-\frac{L_1}{2}\cos(\theta)\dot{\theta}^2 - \frac{L_1}{2}\sin(\theta)\ddot{\theta}\right) = F_{R,2}\cos(\theta) - \frac{2T}{L_1}\sin(\theta) - F_{R,1}\sin(\theta) - F_{R,3}\sin(\theta)$$

$$M\left(-\frac{L_1}{2}\sin(\theta)\dot{\theta}^2 + \frac{L_1}{2}\cos(\theta)\ddot{\theta}\right) = F_{R,2}\sin(\theta) + \frac{2T}{L_1}\cos(\theta) + F_{R,1}\cos(\theta) + F_{R,3}\cos(\theta)$$

(b) The horizontal and vertical Newton's laws for the beam (at its centre of mass) are given as follows:

$$m\ddot{x}_2 = F\cos(\Theta) + F_{R,3}\sin(\Theta)$$

$$m\ddot{y}_2 = F\sin(\Theta) - F_{R,3}\cos(\Theta)$$

Substituting $\ddot{x}_2 = \ddot{r}\cos(\theta) - 2\sin(\theta)\dot{r}\dot{\theta} - \left(r + \frac{L_1}{2}\right)\cos(\theta)\dot{\theta}^2 - \left(r + \frac{L_1}{2}\right)\sin(\theta)\ddot{\theta}$ and

$\ddot{y}_1 = \ddot{r}\sin(\theta) + 2\cos(\theta)\dot{r}\dot{\theta} - \left(r + \frac{L_1}{2}\right)\sin(\theta)\dot{\theta}^2 + \left(r + \frac{L_1}{2}\right)\cos(\theta)\ddot{\theta}$ to above equations,

we get

$$m\left(\ddot{r}\cos(\theta) - 2\sin(\theta)\dot{r}\dot{\theta} - \left(r + \frac{L_1}{2}\right)\cos(\theta)\dot{\theta}^2 - \left(r + \frac{L_1}{2}\right)\sin(\theta)\ddot{\theta}\right) = F\cos(\theta) + F_{R,3}\sin(\theta)$$

$$m\left(\ddot{r}\sin(\theta) + 2\cos(\theta)\dot{r}\dot{\theta} - \left(r + \frac{L_1}{2}\right)\sin(\theta)\dot{\theta}^2 + \left(r + \frac{L_1}{2}\right)\cos(\theta)\ddot{\theta}\right) = F\sin(\theta) - F_{R,3}\cos(\theta)$$

(c) The rotational Newton's law for the cylinder is:

$$J_1\ddot{\theta} = F_{R,3}\frac{L_1}{2} - F_{R,1}\frac{L_1}{2}$$

The rotational Newton's law for the beam is

$$J_2\ddot{\theta} = -F_{R,3}\left(\frac{L_1}{2} - r\right)$$

(d) Let us compute the first equation of motion.
Multiplying both sides of the horizontal Newton's law of **the cylinder** by -sin (θ) and multiplying both sides of the vertical Newton's law of **the cylinder** by cos(θ), we get:

$$M\left(\frac{L_1}{2}\sin(\theta)\cos(\theta)\dot{\theta}^2+\frac{L_1}{2}\sin^2(\theta)\ddot{\theta}\right)=-F_{R,2}\sin(\theta)\cos(\theta)$$
$$+\frac{2T}{L_1}\sin^2(\theta)+F_{R,1}\sin^2(\theta)+F_{R,3}\sin^2(\theta)$$
$$M\left(-\frac{L_1}{2}\sin(\theta)\cos(\theta)\dot{\theta}^2+\frac{L_1}{2}\cos^2(\theta)\ddot{\theta}\right)=F_{R,2}\sin(\theta)\cos(\theta)$$
$$+\frac{2T}{L_1}\cos^2(\theta)+F_{R,1}\cos^2(\theta)+F_{R,3}\cos^2(\theta)$$

Adding both equations, we get

$$M\left(\frac{L_1}{2}\ddot{\theta}\right)=\frac{2T}{L_1}+F_{R,1}+F_{R,3}$$

(A)

Multiplying both sides of the horizontal Newton's law of **the beam** by -sin (θ) and multiplying both sides of the vertical Newton's law of **the beam** by cos(θ), we obtain

$$m\left(-\ddot{r}\sin(\theta)\cos(\theta)+2\sin^2(\theta)\dot{r}\dot{\theta}+\left(r+\frac{L_1}{2}\right)\sin(\theta)\cos(\theta)\dot{\theta}^2+\left(r+\frac{L_1}{2}\right)\sin^2(\theta)\ddot{\theta}\right)=$$
$$-F\sin(\theta)\cos(\theta)-F_{R,3}\sin^2(\theta)$$
$$m\left(\ddot{r}\sin(\theta)\cos(\theta)+2\cos^2(\theta)\dot{r}\dot{\theta}-\left(r+\frac{L_1}{2}\right)\sin(\theta)\cos(\theta)\dot{\theta}^2+\left(r+\frac{L_1}{2}\right)\cos^2(\theta)\ddot{\theta}\right)=$$
$$F\sin(\theta)\cos(\theta)-F_{R,3}\cos^2(\theta)$$

Adding both equations, we get

$$m\left(2\dot{r}\dot{\theta}+\left(r+\frac{L_1}{2}\right)\ddot{\theta}\right)=-F_{R,3} \qquad \textbf{(B)}$$

Adding the two rotational Newton's law for **the cylinder** and **the beam**, we get

$$(J_1+J_2)\ddot{\theta}=F_{R,3}r-F_{R,1}\frac{L_1}{2}$$
$$\Leftrightarrow F_{R,1}\frac{L_1}{2}=F_{R,3}r-(J_1+J_2)\ddot{\theta} \qquad \textbf{(C)}$$

Substituting **(C)** to **(A)**, we get

$$M\left(\frac{L_1^2}{4}\ddot{\theta}\right)=T-(J_1+J_2)\ddot{\theta}+\left(r+\frac{L_1}{2}\right)F_{R,3} \qquad \textbf{(D)}$$

Substituting **(B)** to **(D)**, we get

$$M\left(\frac{L_1^2}{4}\ddot{\theta}\right)=T-(J_1+J_2)\ddot{\theta}-\left(r+\frac{L_1}{2}\right)m\left(2\dot{r}\dot{\theta}+\left(r+\frac{L_1}{2}\right)\ddot{\theta}\right)$$
$$\Leftrightarrow\left(M\frac{L_1^2}{4}+J_1+J_2+\left(r+\frac{L_1}{2}\right)^2\right)\ddot{\theta}+2m\left(r+\frac{L_1}{2}\right)\dot{r}\dot{\theta}=T$$

Now we will compute the second equation of motion:
Multiplying both sides of the horizontal Newton's law of **the beam** by cos(θ) and by multiplying both sides of the vertical Newton's law of **the beam** by sin(θ), we get

$$m\left(\ddot{r}\cos^2(\theta) - 2\cos(\theta)\sin(\theta)\dot{r}\dot{\theta} - \left(r + \frac{L_1}{2}\right)\cos^2(\theta)\dot{\theta}^2 - \left(r + \frac{L_1}{2}\right)\cos(\theta)\sin(\theta)\ddot{\theta}\right) =$$
$$F\cos^2(\theta) + F_{R,3}\cos(\theta)\sin(\theta)$$

$$m\left(\ddot{r}\sin^2(\theta) + 2\cos(\theta)\sin(\theta)\dot{r}\dot{\theta} - \left(r + \frac{L_1}{2}\right)\sin^2(\theta)\dot{\theta}^2 + \left(r + \frac{L_1}{2}\right)\cos(\theta)\sin(\theta)\ddot{\theta}\right) =$$
$$F\sin^2(\theta) - F_{R,3}\cos(\theta)\sin(\theta)$$

Adding these two equations, we get the second equation of motion:

$$m\left(\ddot{r} - \left(r + \frac{L_1}{2}\right)\dot{\theta}^2\right) = F$$